Author's Solution for the cash award question – 01.08.2023

Our Author is known for his distinct style of solving the Geometric problems. Let's see below how novelly he handles the concerns and solves this problem.

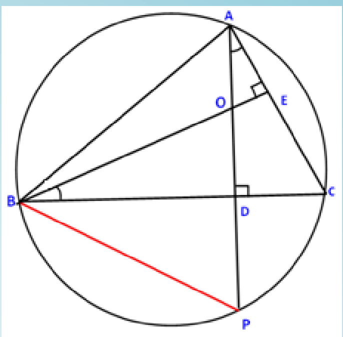
Construction:

Join OT & OS. Let the meeting point of PQ & RC be G.

Solution:

Result: Let us see an important result, before going to solution See box below.

Δ ABC is inscribed in a circle. AD, the altitude to BC is produced to meet the circle at P. BE is the altitude to CA and O is the Orthocenter. Prove that OD=DP.



To prove:-

OD = DP

Construction:-

Join BP

Proof:-

$\angle PBC = \angle PAC$ (Angles in the same segment)……(1)

ABDE is concyclic [$\because \angle ADB = \angle AEB = 90^{\circ}$]

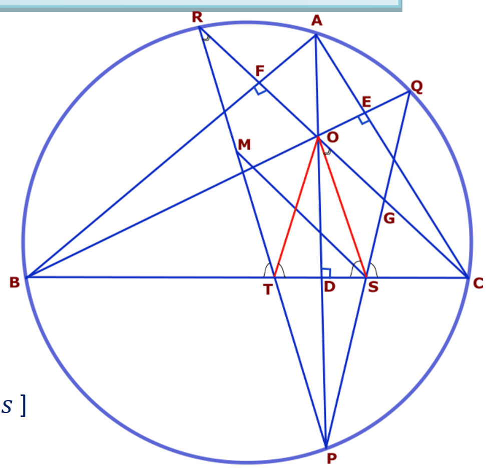
$\Longrightarrow \angle DBE = \angle DAE$ ……………………..(2)

(1) & (2) $\rightarrow \angle PBD = \angle OBD$

∴BD is the perpendicular bisector of OP.

OD = DP ---------- ***Proved.***

(This result is available in page no. 15 of the author's book "The Novelties of Geometry")



See Picture:

Vide the above result we have proved OD=DP

∴ TDS is the perpendicular bisector of OP

∴ OT = TP & OS = SP

$\angle ABQ = \angle APQ$ -------------(1) [Angles in the same segment]

$\angle APR = \angle ACR$ -------------(2) [Angles in the same segment]

$\angle OBF = \angle OCE$ ------------- [FBCE is concyclic, as $\angle F$ & $\angle E$ $are\ right\ angles$]

ie $\angle ABQ = \angle ACR$ ----------------------(3)

(1), (2) & (3) →

$\angle APR = \angle APQ$

$\Rightarrow$ PO is the angle bisector of $\angle TPS$

$\therefore$ TD = DS   ($\because OP \perp TS$)

$\Rightarrow$ OP & TS are diagonals of Quadrilateral OTPS and they are perpendicular bisectors for each other. Therefore TOSP is a Rhombus

$\therefore$ OT = OS = PS = PT

And SO ∥ TP

Now, consider ΔOTR & ΔGSO.

$\angle SOG = \angle TRO$ ------------------ (4)   [SO ∥ TP or RP]

$\angle OTR = 180° - \angle OTD - \angle RTB$ ---------(5)

$\angle GSO = 180° - \angle OST - \angle GSC$ ---------- (6)

$\angle GSC = \angle RTB$ ---------------- (7) [$\because \angle PTS = \angle PST$]

And $\angle OTD = \angle OSD$ ---------(8)

(5),(6),(7) & (8) →

$\angle OTR = \angle GSO$ -----------------------------(9)

(4) & (9) →

ΔOTR & ΔGSO are similar

$\therefore \frac{OT}{GS} = \frac{TR}{SO}$

$\Rightarrow \frac{PS}{GS} = \frac{TR}{TP}$ -------------------- (10)   [OT=PS & TP = SO]

But $SM \parallel GR$ [Given]

$\Rightarrow \frac{PS}{SG} = \frac{PM}{MR}$ ----------------- (11)

(10) & (11) →

$\frac{TR}{TP} = \frac{PM}{MR}$

$\Rightarrow \frac{PR-TP}{TP} = \frac{PR-MR}{MR}$

$= \frac{PR}{TP} - 1 = \frac{PR}{MR} - 1$

ie $\frac{PR}{TP} = \frac{PR}{MR}$

$\therefore TP = MR$ -------------------Proved

****